Chapter 5
Strictly Bandlimited Modulations with Large Envelope Fluctuation (Nyquist Signaling)

Nyquist signaling schemes, which by the very nature of their construction are strictly bandlimited, clearly result in the most bandwidth-efficient modulations of all the ones considered previously in this monograph; however, they also result in modulations with the largest envelope fluctuation. Since the theory of Nyquist signaling is well documented in many textbooks on digital communications, e.g., [1–3], we shall present here only a brief summary of the basic principles simply as a matter of completeness. Although most of the discussion will be focussed on single-channel binary signaling, the extension to multilevel and quadrature signaling schemes such as QAM will be immediately obvious and will receive a brief treatment.

5.1 Binary Nyquist Signaling

The Nyquist criterion is a condition imposed on a waveform that results in zero ISI when a sequence of such waveforms amplitude-modulated by the data is sequentially transmitted at a fixed data rate. Specifically, a binary Nyquist signal is one whose underlying pulse shape, $p(t)$, has uniform samples taken at the bit rate, $1/T_b$ (i.e., herein referred to as the Nyquist rate), that satisfy

$$p_n = p(nT_b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)\, e^{j\omega n T_b} d\omega = \delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{5.1-1}$$

Since the Nyquist criterion is derived based on the sampling theorem, the signals to which it is applied are inherently strictly bandlimited. To see this, we proceed as follows:

The integral in (5.1-1) can be written in terms of a partition of adjacent radian frequency intervals of width $2\pi\,(1/T_b) = 2\pi\,(2W)$, viz.,

$$p_n = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{(\pi/T_b)(2k-1)}^{(\pi/T_b)(2k+1)} P(\omega)\, e^{j\omega n T_b} d\omega = \delta_n \tag{5.1-2}$$

Using the change of variables $v = \omega - 2k\pi/T_b$, (5.1-2) becomes

$$\begin{aligned} p_n &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-(\pi/T_b)}^{(\pi/T_b)} P\left(v + \frac{2k\pi}{T_b}\right) e^{jnT_b(v+[2k\pi/T_b])} dv \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-(\pi/T_b)}^{(\pi/T_b)} P\left(v + \frac{2k\pi}{T_b}\right) e^{jvnT_b} dv \\ &= \frac{1}{2\pi} \int_{-(\pi/T_b)}^{(\pi/T_b)} \sum_{k=-\infty}^{\infty} P\left(v + \frac{2k\pi}{T_b}\right) e^{jvnT_b} dv \end{aligned} \tag{5.1-3}$$

Next, define the equivalent Nyquist channel characteristic

$$P_{eq}(\omega) = \begin{cases} \sum\limits_{k=-\infty}^{\infty} P\left(\omega + \dfrac{2k\pi}{T_b}\right), & |\omega| \le \dfrac{\pi}{T_b} \\ 0, & \text{otherwise} \end{cases} \tag{5.1-4}$$

i.e., all of the translates of $P(\omega)$ folded into the interval $(-\pi/T_b, \pi/T_b)$ and superimposed on each other. Substituting (5.1-4) into (5.1-3) gives

$$p_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{eq}(\omega)\, e^{j\omega n T_b} d\omega \tag{5.1-5}$$

But the inverse Fourier transform of $P_{eq}(\omega)$ is, by definition,

$$p_{eq}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{eq}(\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-(\pi/T_b)}^{(\pi/T_b)} P_{eq}(\omega)\, e^{j\omega t} d\omega \tag{5.1-6}$$

Thus, from (5.1-5) and (5.1-6), we see that the Nyquist rate samples of $p(t)$, namely, p_n, are also the Nyquist rate samples of $p_{eq}(t)$. Since, by the definition of (5.1-4), $P_{eq}(\omega)$ is a strictly bandlimited function on the interval $(-\pi/T_b, \pi/T_b) = (-2\pi W, 2\pi W)$, then, from the sampling theorem,

$$P_{eq}(\omega) = \begin{cases} \dfrac{1}{2W} \sum\limits_{n=-\infty}^{\infty} p\left(\dfrac{n}{2W}\right) \exp\left(-jn\dfrac{\omega}{2W}\right), & |\omega| \le 2\pi W \\ 0, & \text{otherwise} \end{cases} \tag{5.1-7a}$$

or equivalently

$$P_{eq}(\omega) = \begin{cases} T_b \sum\limits_{n=-\infty}^{\infty} p_n \exp(-jn\omega T_b), & |\omega| \le \dfrac{\pi}{T_b} \\ 0, & \text{otherwise} \end{cases} \tag{5.1-7b}$$

However, since for zero ISI we require $p_n = \delta_n$, then (5.1-7b) simplifies to

$$P_{eq}(\omega) = \begin{cases} T_b, & |\omega| \le \dfrac{\pi}{T_b} \\ 0, & \text{otherwise} \end{cases} \tag{5.1-8}$$

i.e., the equivalent Nyquist channel characteristic is an ideal brick wall filter. Finally, combining (5.1-4) and (5.1-8), we see that the Nyquist channel (Fourier transform of the Nyquist pulse) $P(\omega)$ must satisfy

$$\sum_{k=-\infty}^{\infty} P\left(\omega + \frac{2k\pi}{T_b}\right) = T_b, \quad |\omega| \le \frac{\pi}{T_b} \tag{5.1-9}$$

i.e., the superposition of all the translates of $P(\omega)$ must yield a flat spectrum in the Nyquist bandwidth $(-\pi/T_b, \pi/T_b)$. It can also be shown that the superposition of all the translates of $P(\omega)$ must yield a flat spectrum in the interval $((2k-1)\pi/T_b, (2k+1)\pi/T_b)$ for any k. Thus, combining the equation that would result from this fact with (5.1-9) gives

$$\sum_{k=-\infty}^{\infty} P\left(\omega + \frac{2k\pi}{T_b}\right) = T_b \tag{5.1-10}$$

for all ω. Note that the zero ISI criterion does not uniquely specify the pulse shape spectrum $P(\omega)$ unless its bandwidth happens to be limited to $(-\pi/T_b, \pi/T_b)$, in which case, it must itself be flat, since the sum in (5.1-10) reduces to a single term, namely, $k = 0$. The implication of this statement is (as we shall soon see) that there are many $P(\omega)$'s that satisfy the zero ISI condition.

Consider now a system transmitting a baseband signal of the form $s(t) = \sqrt{P}\sum_{n=-\infty}^{\infty} a_n p(t - nT_b)$ where $p(t)$ satisfies the Nyquist condition and $\{a_n\}$ are binary (± 1) symbols. Then, based on the above, the minimum lowpass, single-sided bandwidth needed to transmit this signal at rate $R = 1/T_b$ without ISI is $R/2 = 1/2T_b$. Such transmissions occur when the equivalent channel $P_{eq}(\omega)$ has a rectangular transfer function or equivalently

$$p_{eq}(t) = p(t) = \frac{\sin \dfrac{\pi t}{T_b}}{\dfrac{\pi t}{T_b}} \tag{5.1-11}$$

When the binary symbols are independent and the noise samples (spaced T_b s apart) are uncorrelated, each symbol can be recovered without resorting to past history of the waveform, i.e., with a zero memory receiver.

Since, in the above case, R b/s are transmitted without ISI over a baseband bandwidth $R/2$ hertz, then the throughput efficiency is R (b/s)/($R/2$) hertz = 2 (b/s)/hertz. To achieve this efficiency, one must generate the sin x/x pulse shape of (5.1-11), which, in theory, is a noncausal function and extends from $-\infty$ to ∞. This pulse shape is additionally impractical because of its very slowly decreasing tail, which will cause excessive ISI if any perturbations from the ideal sampling instants should occur. Stated another way, the price paid for the extreme bandwidth efficiency achieved with this Nyquist pulse is a large variation in the instantaneous amplitude of the pulse, resulting in a high sensitivity to timing (sampling instant) offset.

To reduce this sensitivity, one employs more practical shapes for $p(t)$, whose Fourier transforms, $P(\omega)$, have smoother transitions at the edges of the band, yet still satisfy the Nyquist condition, thereby resulting in zero ISI. As a consequence, these waveforms will not achieve the minimum Nyquist bandwidth, as we shall see momentarily. The raised cosine transfer function

$$P(\omega) = \begin{cases} T_b, & 0 \le |\omega| \le \dfrac{\pi}{T_b}(1-\alpha) \\[2ex] T\cos^2\left\{\dfrac{\pi}{4\alpha}\left[\dfrac{|\omega| T_b}{\pi} - 1 + \alpha\right]\right\}, & \dfrac{\pi}{T_b}(1-\alpha) \le |\omega| \le \dfrac{\pi}{T_b}(1+\alpha) \\[2ex] 0, & \dfrac{\pi}{T_b}(1+\alpha) \le |\omega| \le \infty \end{cases} \tag{5.1-12}$$

with excess bandwidth $\alpha R/2\,(0 \leq \alpha \leq 1)$ (see Fig. 5-1a) satisfies the Nyquist criterion and has a pulse shape whose tails decrease faster than the sin x/x function, i.e., they are the product of sin x/x and $\cos\left(\pi\alpha t/T_b\right)/[1-\left(2\alpha t/T_b\right)^2]$ [see Fig. 5-1(b)]. Note that these pulses are still noncausal and extend from $-\infty$ to ∞—properties that are a direct consequence of the strict bandlimitation of the Nyquist formulation. Since the bandwidth of this class of Nyquist pulses is $R/2\,(1+\alpha)$, the price paid for improved sensitivity to timing jitter is a reduction of the throughput efficiency to $R/\left[R/2\ (1+\alpha)\right] = 2/\left(1+\alpha\right)$. Ideally (perfect sampling), the error probability of all binary Nyquist signaling schemes is equivalent to that of ideal binary PSK, as given by (2.6-2).

5.2 Multilevel and Quadrature Nyquist Signaling

To achieve higher throughput efficiencies, one can extend the above notions to multilevel and quadrature signaling schemes. First, since the Nyquist criterion does not impact the choice of levels for the data symbols, one may simply employ an M-ary alphabet for $\{a_n\}$, e.g., $a_n = \pm 1, \pm 3, \cdots, \pm\left(M-1\right)$, resulting in a form of M-ary pulse amplitude modulation. Using the raised cosine Nyquist pulse of (5.1-12), the throughput efficiency is increased to $2\log_2 M/\left(1+\alpha\right)$. If now one modulates independent Nyquist signals on I and Q carriers, resulting in a form of pulse-shaped M^2-QAM results, the throughput is further increased to $4\log_2 M/\left(1+\alpha\right)$. Of course, if one specifically chooses $M = 4$, what results is Nyquist-pulse-shaped QPSK.

References

[1] M. K. Simon, S. M. Hinedi, and W. C. Lindsey, *Digital Communication Techniques: Signal Design and Detection*, Upper Saddle River, New Jersey: Prentice Hall, 1995.

[2] J. Proakis, *Digital Communications*, 3rd edition, New York: McGraw-Hill, 1995.

[3] E. A. Lee and D. G. Messerschmitt, *Digital Communication*, 2nd edition, Boston, Massachusetts: Kluwer Academic Publishers, 1994.

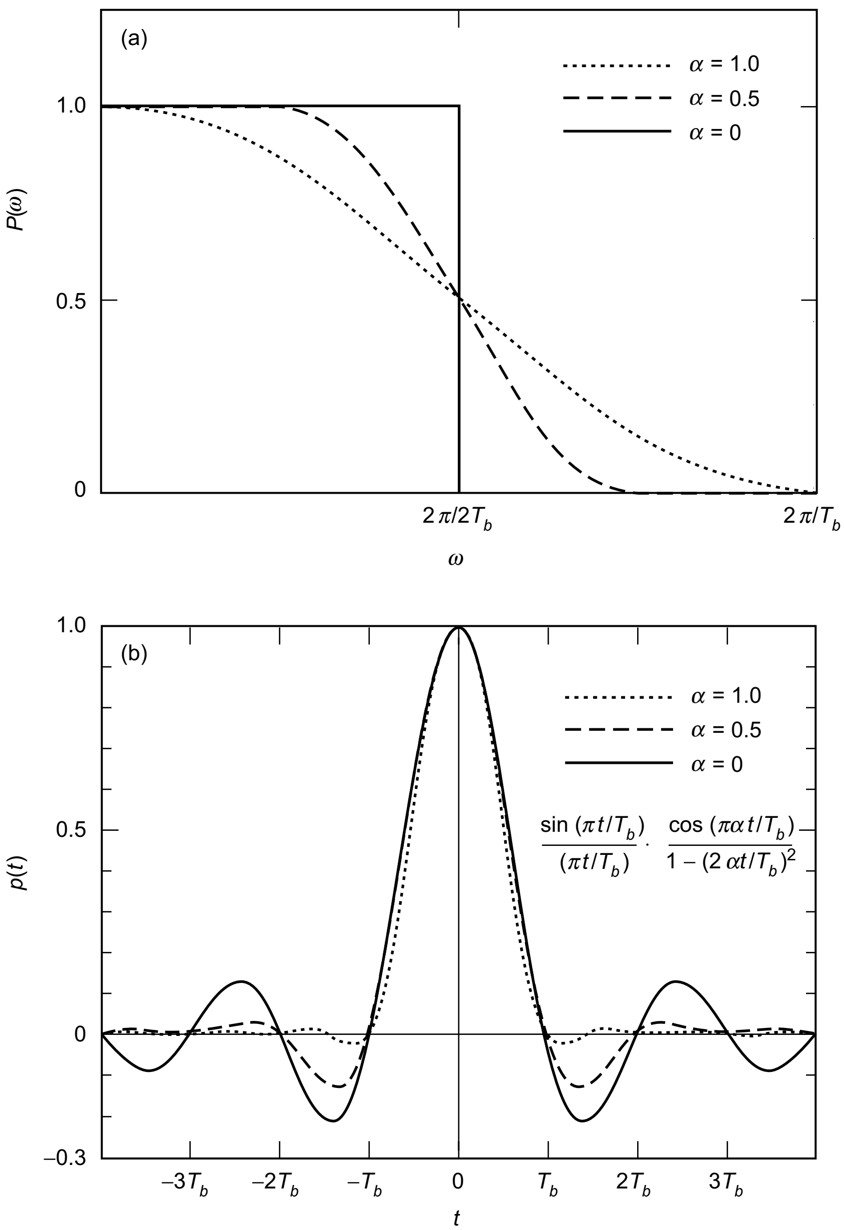

Fig. 5-1. The raised cosine pulse: (a) frequency function and (b) time function.